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Geometric bounds on the linearization domain and analytic dependence on parameters for families of analytic vector fields in a neighborhood of a singular point

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ABSTRACT

We study families of holomorphic vector fields, holomorphically depending on parameters, in a neighborhood of an isolated singular point. When the singular point is in the Poincaré domain for every vector field of the family we prove, through a modification of classical Sternberg's linearization argument, cf. Nelson (1969) [7] too, analytic dependence on parameters of the linearizing maps and geometric bounds on the linearization domain: each vector field of the family is linearizable inside the smallest Euclidean sphere which is not transverse to the vector field, cf. Brushlinskaya (1971) [2], Ilyashenko and Yakovenko (2008) [5] for related results. We also prove, developing ideas in Martinet (1980) [6], a version of Brjuno's Theorem in the case of linearization of families of vector fields near a singular point of Siegel type, and apply it to study some 1-parameter families of vector fields in two dimensions.

1. Introduction

Let $U \subset \mathbb{C}^n$ be a neighborhood of the point O and let

$$\mathbb{X} : U \times \Omega \to T'\mathbb{C}^n$$

be a holomorphic family of vector fields depending on parameters $\eta \in \Omega \subset \mathbb{C}^p$. We suppose that all the vector fields $\mathbb{X}_\eta$ of the family have a singular point at O: hence $\mathbb{X}$ is, roughly speaking, a p-manifold in

$$\chi = \chi(U) = \{\text{holomorphic vector fields in } U \text{ singular at } O\}.$$

Let $\chi_m = j^m\chi$ be the finite-dimensional vector space of m-jets at O of holomorphic vector fields in χ. Each $\mathbb{X} \in \chi$, and $j^m\mathbb{X} \in \chi_m$ too, is a derivation, therefore $j^m\mathbb{X} \in Hom(\chi_m, \chi_m)$ and it has a Jordan Normal Form $j^m\mathbb{X} = j^m\mathbb{S} + j^m\mathbb{N}$, where $j^m\mathbb{S}$ is semi-simple, $j^m\mathbb{N}$ is nilpotent and $[j^m\mathbb{S}, j^m\mathbb{N}] = 0$. These decompositions at different levels m are compatible, i.e.

$$j^m\bigl(j^{m+1}\mathbb{X}\bigr) = j^m\mathbb{X}$$

then they define the Jordan decomposition in χ:

$$\mathbb{X} = \mathbb{S} + \mathbb{N}$$

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where $[\mathbb{S}, \mathbb{N}] = 0$, $\mathbb{S}$ is semi-simple and $\mathbb{N}$ is nilpotent. There exist good (canonical) coordinates z such that $\mathbb{S}$ in these coordinates has the form (see [6]):

$$Sz = \lambda_1 z_1 \frac{\partial}{\partial z_1} + \cdots + \lambda_n z_n \frac{\partial}{\partial z_n}. \tag{1}$$

A *normal form* of $\mathbb{X} = \mathbb{S} + \mathbb{N}$ is any expression in z-coordinates $X(z) = Sz + N(z)$ of this vector field such that the semi-simple part $\mathbb{S}$ has the expression (1): this rigorous definition given in [6] captures the empiric meaning of the normal form of a vector field, which asks for the "simplest" choice of the coefficients of a vector field up to local changes of coordinates. If in z-coordinates $N(z) = Bz$ is a linear function then we say that $\mathbb{X}$ is *linearizable* and its expression in z coordinates is $\mathbb{X}(z) = D\mathbb{X}(O)z = Sz + Bz$, where $D\mathbb{X}(O)$ is the differential at O of $\mathbb{X}$.

In this article we deal with linearization of families of vector fields. In more detail, we try to get estimates, possibly based on the geometry of the foliations induced by the vector fields of the family, on a common domain of convergence of the linearizing diffeomorphisms and to prove their holomorphic dependence on the parameters. In the second section we consider the simpler case of families of vector fields with a singular point of Poincaré type. In this case we improve some known results in [2,5], showing that in absence of resonances the linearization domain contains the smallest sphere (referred to the standard Euclidean structure in $\mathbb{C}^n$ and to the canonical coordinates putting the semi-simple part of the vector field in the diagonal form (1)) which is not transverse to the foliation defined by the vector field: this estimate gives a uniform geometric bound on the common linearization domain of a family of vector fields, and the holomorphic dependence of the linearizing diffeomorphisms on the parameter follows easily. This result improves the one in [4]: as that article, it relies on a Hurwitz-type lemma on global existence of a diffeomorphism which is limit of a sequence of holomorphic diffeomorphisms, whose proof is here considerably simplified. We mention that similar results, under stronger hypotheses, are proved in the case of normalization of families with a singular point of Poincaré type in [11].

In the last section we deal with the case of a family of vector fields with a singular point at O of Siegel type. In this case is well known that if no resonance relation holds and if the eigenvalue of the vector field satisfies an arithmetic condition introduced by Brjuno [3] then linearization is possible: we adapt Brjuno's arithmetic condition to families of vector fields and prove a result of existence of the linearizing diffeomorphisms in a common domain: their analytic dependence on parameters follows easily. The proof of this parameter-depending version of Brjuno's Theorem develops a sketched proof by J. Martinet [6]. We apply this result to the case of a family of vector fields in $\mathbb{C}^2$.

2. Geometric bounds on the linearization domain for analytic families of vector fields near a Poincaré singular point

Let $U \subset \mathbb{C}^n$ be a neighborhood of O, $\Omega \subset \mathbb{C}^p$; herein we will always refer to fixed coordinates η in Ω. Let

$$\mathbb{X} : U \times \Omega \to T\mathbb{C}^n$$

be an analytic family of holomorphic vector fields

$$\mathbb{X}_\eta : U \to T\mathbb{C}^n.$$

We suppose that O is an isolated singular point for all the vector fields of the family and denote $D\mathbb{X}_\eta(O)$ the differential at O of $\mathbb{X}_\eta$. Let

$$\operatorname{spec} D\mathbb{X}_\eta(O) = \{\lambda_1(\eta), \ldots, \lambda_n(\eta)\}$$

and

$$\underline{\lambda}(\eta) = (\lambda_1(\eta), \ldots, \lambda_n(\eta)).$$

Let $\overline{\operatorname{co}\{\lambda_1(\eta), \ldots, \lambda_n(\eta)\}}$ be the closure of the convex hull of $\{\lambda_1(\eta), \ldots, \lambda_n(\eta)\}$. The following definition was introduced by Poincaré in his thesis [9]:

Definition 2.1. The n-type $\{\lambda_1(\eta), \ldots, \lambda_n(\eta)\}$ is in the *Poincaré domain* if $0 \notin \overline{\operatorname{co}\{\lambda_1(\eta), \ldots, \lambda_n(\eta)\}}$. We will say that $\mathbb{X}_\eta$ has at O a singular point of Poincaré type if $\operatorname{spec} D\mathbb{X}_\eta(O)$ is in the Poincaré domain.

A choice of z-coordinates put $D\mathbb{X}_\eta(O)$ in *ε-Jordan Normal Form* if:

$$D\mathbb{X}_\eta(O)z = A_\eta z = S_\eta z + \varepsilon B_\eta z \tag{2}$$

where B_η is the nilpotent matrix having entries satisfying $b_{i,j} = 0$ if $i \neq j+1$, and $b_{j+1,j} \in \{0, 1\}$, and

$$S_\eta z = \lambda_1(\eta) z_1 \frac{\partial z_1}{\partial z_1} + \cdots + \lambda_n(\eta) z_n \frac{\partial z_n}{\partial z_n}. \tag{3}$$

Throughout this section the following hypothesis will hold true:

Hypothesis (H). *There exists a coordinate system* (z,η) *in* $U \times \Omega$ *such that* $D\mathbb{X}_\eta(O)z = A_\eta z$ *is in the* ε*-Jordan Normal Form* (2).

Of course, the above hypothesis is by no means restrictive, as the ε-Jordan Normal Form can be obtained by a family of linear transformations analytically depending on $\eta \in \Omega$. We also remark that any linearizing transformation of the type introduced in this section preserves hypothesis (H), being tangent to the identity at O.

Let K be a neighborhood of $O \in \mathbb{C}^p$, $K \Subset \Omega$. Up to multiplication of the vector fields $\mathbb{X}_\eta$ by suitable $\mu = \mu(\eta)$, $|\mu| = 1$, there exist $\alpha, \beta > 0$ such that for every $\eta \in \overline{K}$, denoting $\Re z$ the real part of the complex number z:

$$-\beta \leqslant \Re\lambda_n(\eta) \leqslant \cdots \leqslant \Re\lambda_1(\eta) \leqslant -\alpha < 0. \tag{4}$$

Vector fields satisfying (4) form a *scaled* family in $\overline{K}$: herein we will always suppose to deal with scaled family, i.e. we write $\mathbb{X}_\eta$ for $\mu\mathbb{X}_\eta$, the general case being easily deducible from this one.

We recall, see the next section for more details, that a *resonance* is a relation:

$$\underline{\lambda} \cdot \underline{m} - \lambda_j = 0$$

where $\underline{m} = (m_1, \ldots, m_n)$, $m_j \in \mathbb{N}_0$, $|\underline{m}| = m_1 + \cdots + m_n \geqslant 2$ and $j = 1, \ldots, n$. If $\{\lambda_1, \ldots, \lambda_n\}$ is in the Poincaré domain then

$$|\underline{m}| \leqslant \frac{\beta}{\alpha}.$$

In the ε-Jordan Normal Form of $D\mathbb{X}_\eta(O)$ we will consider $\varepsilon < \frac{\alpha}{2}$.

Let $\mathcal{F}_\eta$ be the holomorphic foliation by curves of U defined by $\mathbb{X}_\eta$: from [1] the fact that $\mathbb{X}_\eta$ is in the Poincaré domain is equivalent to the geometric property

$$\mathcal{F}_\eta \pitchfork S_r \tag{5}$$

where S_r is any Euclidean sphere of radius r, with respect to the standard metric on $\mathbb{C}^n$ defined by the $z = z(\eta)$-coordinates of the ε-Jordan Normal Form, and $r < r_0(\eta)$, where

$$r_0(\eta) = \inf\{r > 0\colon\ \mathcal{F}_\eta \pitchfork S_r \text{ is false}\}. \tag{6}$$

Analogously we define

$$r_0 = \inf\{r > 0\colon\ \text{there exists } \eta \in K \text{ such that } \mathcal{F}_\eta \pitchfork S_r \text{ is false}\}. \tag{7}$$

From the fact that transversality is an open condition and from compactness of $\overline{K}$ it follows that $r_0 > 0$.

We can summarize the last remarks stating the following property (S) which follows from (5):

Property (S). *There exists* $r_0 > 0$ *such that for every* $\eta \in K$:

$$\mathcal{F}_\eta \pitchfork S_r$$

for every $r < r_0$, *where* S_r *is the Euclidean sphere with respect to coordinates* $z(\eta)$.

We can now state a quantitative version of (5), see also [4]: we denote $|z|$ the standard norm of $z \in \mathbb{C}^n$ and ϕ^t_η the flow of $\mathbb{X}_\eta$. The following lemma is an exponential stability result of O for a family of vector fields having a singular point of Poincaré type.

Lemma 2.1. *Let* $\mathbb{X}_\eta$, $\eta \in K$ *be an analytic family of vector fields. Then there exists* $\varepsilon_0 > 0$ *such that for every* $r \in (0, r_0(\eta))$ *and* $\varepsilon \in (0, \varepsilon_0)$ *there exists* $\delta = \delta(r,\eta)$ *such that* $\delta \to 0$ *if* $r \to 0$ *and*

$$\left|\phi^t_\eta(z)\right| < e^{(-\alpha+\varepsilon+\delta)t}|z| \tag{8}$$

for every $t > 0$, $|z| < r$ *and* $\eta \in K$, *and* α *has been defined in* (4). *Moreover if* $r_0(\eta)$ *defined in* (6) *is substituted by* r_0 *defined in* (7) *then there exists* $\delta = \delta(r)$, *independent of* $\eta \in K$, *such that* $\delta = \delta(r_0)$, $\delta \to 0$ *as* $r_0 \to 0$ *and* (8) *holds for every* $t > 0$, $|z| < r$, $r < r_0$, $\eta \in K$.

Proof. Firstly we suppose that $\eta \in K$ is fixed, and prove the η-depending part of the statement.

For $t \in \mathbb{R}$ and $z = z(\eta)$ coordinates such that $D\mathbb{X}_\eta(O)$ is in ε-Jordan Normal Form, we consider the differential equations with real independent variable ("time"):

$$\frac{dz}{dt} = X_\eta(z) = S_\eta z + \varepsilon B_\eta z + N_\eta(z)$$

where $N_\eta(z) = \mathcal{O}(|z|^2)$. Then

$$\frac{d}{dt}\overline{\phi_\eta^t} = \overline{\frac{d}{dt}\phi_\eta^t}$$

where $\bar{z}$ stays for complex conjugate of z, and therefore

$$\frac{d}{dt}\bigg|_t \left|\phi_\eta^t(z)\right|^2 = \sum_{j=1}^{n} 2\Re\left(\frac{d}{dt}\bigg|_t \phi_{\eta,j}^t(z)\overline{\phi_{\eta,j}^t(z)}\right).$$

From

$$\frac{d}{dt}\bigg|_t \phi_{\eta,j}^t(z)\overline{\phi_{\eta,j}^t(z)} = \lambda_j(\eta)\left|\phi_{\eta,j}^t(z)\right|^2 + \varepsilon\left(B_\eta\phi_\eta^t(z)\right)_j\overline{\phi_{\eta,j}^t(z)} + N_{\eta,j}\left(\phi_\eta^t(z)\right)\overline{\phi_{\eta,j}^t(z)}$$

we get:

$$\frac{d}{dt}\bigg|_t \left|\phi_\eta^t(z)\right|^2 = 2\left(\sum_{j=1}^{n} \Re\lambda_j(\eta)\left|\phi_{\eta,j}^t(z)\right|^2 + \varepsilon\Re < B_\eta\phi_\eta^t(z),\ \phi_\eta^t(z) >_{\mathbb{C}^n} + \mathcal{O}\left(\left|\phi_\eta^t(z)\right|^3\right)\right).$$

The definition of B_η implies that:

$$|B_\eta z| \leqslant |z|$$

hence:

$$\left|\Re < B_\eta\phi_\eta^t(z), \phi_\eta^t(z) >_{\mathbb{C}^n}\right| \leqslant \left|\phi_\eta^t\right|^2. \tag{9}$$

From (4), (9):

$$\frac{d}{dt}\bigg|_t \left|\phi_\eta^t(z)\right|^2 \leqslant 2(-\alpha+\varepsilon)|_t\left|\phi_\eta^t(z)\right|^2 + \mathcal{O}\left(\left|\phi_\eta^t(z)\right|^3\right). \tag{10}$$

From (5) for every $\eta \in K$ there exists $\delta = \delta(r,\eta)$, $\delta \to 0$ as $r \to 0$, such that for every $t > 0$:

$$\mathcal{O}\left(\left|\phi_\eta^t(z)\right|^3\right) \leqslant \delta\left|\phi_\eta^t(z)\right|^2. \tag{11}$$

Then from (10), (11):

$$\frac{d}{dt}\bigg|_t \left|\phi_\eta^t(z)\right|^2 \leqslant 2(-\alpha+\varepsilon+\delta)\left|\phi_\eta^t(z)\right|^2$$

from which (8) follows.

The same arguments apply to give a definition of δ independent of $\eta \in K$, hence completing the proof: in fact, the above argument allows to define $\delta(r,\eta)$ which is locally constant in a small neighborhood of η, and compactness of $\overline{K}$ ends the proof. $\square$

The next lemma is a Hurwitz-type result for sequences of holomorphic diffeomorphisms: it is a version of an analogous result in [4], but its proof is here considerably simplified.

Lemma 2.2. *Let D, D' be bounded domains in $\mathbb{C}^n$ and*

$$f_m : D \to D',$$

$m \in \mathbb{N}$, be a sequence of holomorphic maps such that

$$f_m : D \to D'$$

are diffeomorphisms onto their images. Let $\{f_m\}_{m=1}^\infty$ converges uniformly on the compact subsets of D. Then the following dichotomy holds:

- *either* $\det Df \equiv 0$ *in D,*
- *or $f : D \to D'$ is a diffeomorphism onto its image.*

Proof. Firstly we prove that if there exists $z' \in D$ such that $\det Df(z') \neq 0$ then $\det Df(z) \neq 0$ for every $z \in D$. By contradiction, let $z'' \in D$, $\det Df(z'') = 0$. As D is pathwise connected, a simple argument reduces the proof to the case when z', z'' lay on a topological disk $\Delta \subset D$, Δ contained in the complex line l joing z', z''. Let us define on Δ the sequence of functions of a complex variable ξ parameterizing the line l:

$$g_m(\xi) = \det Df_m\big(z(\xi)\big).$$

This sequence converges uniformly on compact subsets of Δ to $g(\xi) = \det Df(z(\xi))$: from standard Hurwitz's theorem $g(\xi) = \det Df(z(\xi))$ is identically zero or is never zero, and we get a contradiction with the definition of z', z''. Hence f is a local diffeomorphisms in D. If it were not a global diffeomorphism then, as f is holomorphic:

$$\deg(f, \Sigma) > 1$$

for a suitable compact subset Σ of D. This is impossible: in fact for all m's but finitely many:

$$\deg(f_m, \Sigma) = 1$$

and $\deg(f_m, \Sigma) = \deg(f, \Sigma)$. $\square$

Now we can state and prove the main result of this section: see also [2,5] for similar results, not containing geometric estimates of the linearization domain, nor explicit analytic definition of the linearizing maps. To keep statements and proofs as simple as possible we consider only the case of linearization of non-resonance families of vector fields. An analogous result about geometric bounds on the normalization domain and analytic dependence of the normalizing transformations on parameters is contained in [11]: in this case for what concerns analytic dependence on the parameters, attention must be paid to non-uniqueness of the normalizing maps, and stronger hypotheses than (H) should be introduced.

Theorem 2.1. *Let the analytic family*

$$\mathbb{X} : D \times \Omega \to T\mathbb{C}^n$$

satisfy hypothesis (H)*, and let O be an isolated singular point of the Poincaré type for every vector field of the family, such that* $\operatorname{spec} D\mathbb{X}_\eta(O)$ *does not satisfy any resonance relations for $\eta \in K$, $K \Subset \Omega$. Let*

$$X_\eta(z) = A_\eta z + N_\eta(z)$$

where $B_{r_0(\eta)}(O) = \{z\colon |z| < r_0(\eta)\}$ is the Euclidean ball defined in (5) *and $A_\eta z = S_\eta + \varepsilon B_\eta$ is the ε-Jordan Normal Form guaranteed by hypothesis* (H)*, with $0 < \varepsilon < \varepsilon_0$, ε_0 defined in Lemma* 2.1*. There exists an analytic family of diffeomorphisms*:

$$S^\eta(\cdot) = S(\cdot, \eta) : B_{r_0(\eta)}(O) \times K \to \mathbb{C}^n, \tag{12}$$

$$S^\eta(z) = w = \lim_{s \to \infty} e^{-sA_\eta} \phi^s_\eta(z) \tag{13}$$

such that for any $\eta \in K$ S^η is the linearizing diffeomorphism:

$$DS^\eta\big(\big(S^\eta\big)^{-1}(w)\big) X_\eta\big(\big(S^\eta\big)^{-1}(w)\big) = A_\eta w \tag{14}$$

where $r_0(\eta)$ has been defined in (6)*. Moreover, the domain of definition of the family contains the cylinder $B_{r_0}(O) \times K$, r_0 independent of η defined in* (7)*.*

Of course, existence of a linearizing diffeomorphism in a small neighborhood of O for any fixed value of η is a classical result by Poincaré: we are going to prove a geometric bound for the convergence domain of the analytic linearizing diffeomorphisms, and their analytic dependence, in the common convergence cylinder $B_{r_0}(O) \times K$, on η.

Proof. Firstly, we fix $\eta \in K$ and prove the bound on the domain of analytic linearizability of $\mathbb{X}_\eta$: the proof follows the argument introduced in [10], see [7] too, combining it with Lemma 2.2.

For any integer $m \geqslant 2$ the Poincaré–Dulac Theorem implies the existence of a diffeomorphism G_η^{-1}, polynomial with respect to w and analytic with respect to η, such that:

$$z = G_\eta(w) = w + g_\eta(w)$$

with $g_\eta(w) = \mathcal{O}(|w|^2)$, $\tilde{X}_\eta(w) = DG_\eta^{-1}(G_\eta(w)) X_{eta}(G_\eta(w))$, with

$$\tilde{X}_\eta(w) = A_\eta w + \tilde{N}_\eta(w)$$

where

$$\tilde{N}_\eta(w) = \mathcal{O}\big(|w|^m\big) \tag{15}$$

and

$$G_\eta : B_{r_1,\eta}(O) \to B_{r_2,\eta}(O).$$

We can choose $r_1 = r_1(\eta)$ sufficiently small for:

$$G_\eta\big(B_{r_1,\eta}(O)\big) \subset B_{r_0(\eta),z,\eta}(O)$$

and

$$B_{r_1,\eta}(O) \subset B_{r_0(\eta),\eta}(O). \tag{16}$$

Of course, analogous uniform inclusions hold, up to reduction of r_1, where $r_0(\eta)$ is substituted with r_0. We observe that in the last inclusion appear two sets which are spheres with respect to different Euclidean structures, but this will not cause any trouble for our argument are based on transversality property.

The absence of resonances implies that G_η is essentially unique, and it depends analytically on η.

From hypothesis (H) and property (S) the diffeomorphisms:

$$L_t^\eta : B_{r_1}(O) \to \mathbb{C}^n, \qquad L_t^\eta(w) = e^{-tA_\eta}\tilde{\phi}_\eta^t(w)$$

where $\tilde{\phi}^t(w)$ is the flow of $\tilde{X}_\eta(w)$, are defined for $t > 0$ and admit the following integral representation:

$$L_t^\eta(w) = w + \int_0^t e^{-sA_\eta}\tilde{N}_\eta\big(\tilde{\phi}_\eta^s(w)\big)\,ds.$$

Referring to $\delta(r_1, \eta)$ defined in Lemma 2.1 let:

$$\gamma = m(\alpha - \varepsilon - \delta) - \beta.$$

As $\delta(r_1, \eta) \to 0$ as $r_1 \to 0$, choosing sufficiently small r_1 we can find m such that $\gamma > 0$. From (15), and from the expression of $\tilde{X}_\eta(w)$, which turns to be transversal to the spheres in w-coordinates of radii $r < r_1$, we get:

$$\big|L_t^\eta(w) - w\big| \leqslant \frac{C}{\gamma}r_1^m$$

hence choosing a sufficiently small $r_1 = r_1(\eta)$:

$$L_t^\eta\big(B_{r_1}(O)\big) \subset B_{r_0(\eta)}(O)$$

for every $t > 0$: for future use let us remark that these definitions could be based on the definition of $\delta(r_1)$ independent of η in Lemma 2.1, and these arguments would lead to analogous claims valid for every $\eta \in K$. From Lemma 2.1 O is asymptotically stable with basin of attraction containing $B_{r_0}(O)$, therefore from (16) and the above quoted transversality of $\tilde{X}_\eta$ to the w-spheres of radii $r < r_1$, for any sufficiently small $\Delta > 0$ and for any fixed $\eta \in \overline{K}$:

$$s_0(\eta) = \max\big\{\tau = \tau(z, \eta)\colon\ z \in \partial B_{r_0(\eta)-\Delta,z}(O),\ \phi_\eta^\tau(z) \in \overline{B_{\frac{r_1}{2},w}(O)},\ \phi_\eta^t(z) \notin \overline{B_{\frac{r_1}{2},w}(O)} \text{ for } 0 < t < \tau\big\}$$

is a well-defined positive number: here the added lower indices z and w in the definitions of the spheres $B_{r_0(\eta)-\Delta,z}(O)$ and $B_{\frac{r_1}{2},w}(O)$ keeps track of the Euclidean structure defining each of these two sets.

The function $\eta \to s_0(\eta)$ is upper semi-continuous: we prove this claim by contradiction. Let $\varepsilon > 0$ be such that $\eta_n \to \eta$ and

$$\limsup_{n\to\infty} s_0(\eta_n) \geqslant s_0(\eta) + 2\varepsilon$$

then for every $n \in \mathbb{N}$ there exists $p_n \in \partial B_{r_0(\eta)-\Delta,z}(O)$ such that $\tilde{\phi}_{\eta_n}^t(p_n) \notin \overline{B_{\frac{r_1}{2},w}(O)}$ for every $t \in [0, s_0(\eta) + \varepsilon]$. We can suppose that $p_n \to p \in \partial B_{r_0(\eta)-\Delta,z}(O)$, and therefore we should have: $\tilde{\phi}_\eta^t(p) \notin \overline{B_{\frac{r_1}{2},w}(O)}$ for every $t \in [0, s_0(\eta) + \varepsilon]$, contradicting the definition of $s_0(\eta)$.

Therefore we can define:

$$s_0 = \max\big\{s_0(\eta)\colon\ \eta \in \overline{K}\big\}.$$

Let

$$S_t^\eta(z) = e^{-s_0 A_\eta} L_\sigma^\eta \circ \phi_\eta^{s_0}(z)$$

where $t = s_0 + \sigma$. Then for every $t > 0$, $\eta \in K$ and from arbitrariness in the choice of $\Delta > 0$:

$$S_t^\eta(B_{r_0,z(O)}) \subset e^{-s_0 A_\eta} B_{r_3,z}(O) \subset B_{e^{\beta s_0} r_{3,z}}(O)$$

hence from Weierstrass Compactness Principle, see [5], and from uniqueness of the formal linearizing transformation:

$$\lim_{t\to\infty} S_t^\eta = S^\eta,$$
$$S^\eta : B_{r_0,z}(O) \to \mathbb{C}^n$$

and (14) easily follows as in [10], see also [7]. To end the proof we observe that from Weierstrass Principle:

$$DS^\eta(O) = \text{identity in } \mathbb{C}^n$$

therefore from Lemma 2.2 S^η is a global diffeomorphism in $B_{r_0(\eta)}(O) = \{z \in \mathbb{C}^n\colon\ |z| < r_0\}$ for every $\eta \in K$. The uniform estimate on the domain of definition of the linearizing diffeomorphisms, all defined in the cylinder $B_{r_0}(O) \times K$, follows from the substitution of $\delta(r_1, \eta)$ with $\delta(r_1)$ in the above arguments. Finally, the analytic dependence on η of the linearization diffeomorphisms follows from the absence of resonances, the consequent uniqueness of the linearizing maps, the analyticity of $S_t^\eta(z)$ for every $t > 0$ and for $(z, \eta) \in B_{r_0}(O) \times K$ and from Weierstrass Compactness Principle. □

3. On the analytic dependence of the linearizing transformations in the Siegel domain

The previous section deals with families of vector fields having a singular point at O of Poincaré type: the present one deals with some cases when the singular point is of Siegel type.

Definition 3.1. Let $\mathbb{X}$ be a vector field analytic in a neighborhood U of O in $\mathbb{C}^n$ and let O be singular for $\mathbb{X}$. Then $\operatorname{spec} D\mathbb{X}(O) = \{\lambda_1, \dots, \lambda_n\}$ is in the Siegel domain if:

$$0 \in \overline{\operatorname{co}\{\lambda_1, \dots, \lambda_n\}}.$$

In this case we will say that O is a singular point of Siegel type for $\mathbb{X}$.

In this section we will consider an analytic family of vector fields:

$$\mathbb{X} : D \times \overline{\Omega} \to T'\mathbb{C}^n$$

where D is a domain in $\mathbb{C}^n$ containing O and $\Omega \subset \mathbb{C}^p$ is the set of parameters, and we suppose that $\mathbb{X}$ is continuous in $D \times \overline{\Omega}$ and holomorphic in $D \times \Omega$. We will always suppose that O is a singular point for $\mathbb{X}_\eta$, $\eta \in \Omega$, and referring to $\operatorname{spec} D\mathbb{X}_\eta(O)\{\lambda_1(\eta), \dots, \lambda_n(\eta)\}$, $\underline{\lambda}(\eta) = (\lambda_1(\eta), \dots, \lambda_n(\eta))$ we define:

$$\alpha(\underline{m}, j, \eta) = \underline{\lambda}(\eta) \cdot \underline{m} - \lambda_j(\eta)$$

where $\underline{m} = (m_1, \dots, m_n)$, $|\underline{m}| = m_1 + \dots + m_n \geqslant 2$, $m_j \in \mathbb{N}_0$, $j = 1, \dots, n$: a *resonance* relation of type $(\underline{m}, j)$ corresponds to $\alpha(\underline{m}, j, \eta) = 0$.

We recall now Brjuno's (ω) condition [3] for a vector field, and we introduce an analogous condition for a family of vector fields. Let us define for $k \in \mathbb{N}$:

$$\omega_k(\eta) = \min\bigl\{\bigl|\alpha(\underline{m}, j, \eta)\bigr|\colon\ j = 1, \dots, n,\ |\underline{m}| \leqslant 2^{k+1}\bigr\}. \tag{17}$$

Definition 3.2. A vector field $\mathbb{X}_\eta$, $\eta \in \Omega$, satisfies Brjuno's (ω, η) condition if:

$$\sum_{k=1}^{\infty} \frac{\log \frac{1}{\omega_k(\eta)}}{2^k} < \infty.$$

The family of vector fields $\mathbb{X}_\eta$, $\eta \in \Omega$ satisfies Brjuno's (ω) condition if there exists a sequence $\{\omega_k\}_{k=1}^\infty$ and a positive constant C such that for every $k \in \mathbb{N}$: $0 < \omega_k \leqslant \omega_k(\eta)$ for every $\eta \in \Omega$, and $\omega_k < C$ and moreover:

$$\sum_{k=1}^{\infty} \frac{\log \frac{1}{\omega_k}}{2^k} < \infty. \tag{18}$$

The next theorem is a version of Brjuno's Linearization Theorem adapted to the case of a family of vector fields, showing the existence of a common convergence domain and the analytic dependence on parameters of the linearizing diffeomorphisms of each vector field of the family. The proof reported here is based on the one sketched by J. Martinet in [6], Theorem 3, for the case of a single vector field. We will state and prove it, as in [6], for formally semi-simple vector fields: a vector field $\mathbb{X} = \mathbb{S} + \mathbb{N}$ is *formally semi-simple* if there exists a formal change of coordinates transforming $\mathbb{X}$ to $\mathbb{S}$: e.g. when there are no resonance relations and $\mathbb{N}$ is nonlinear.

Theorem 3.1. *Let $\mathbb{X} : D \times \overline{\Omega} \to T\mathbb{C}^n$ be continuous, holomorphic in $D \times \Omega$ and formally semi-simple. We suppose that for every $\eta \in \Omega$ the vector field $\mathbb{X}_\eta$ has an isolated singular point at O and that no resonance relations are satisfied; moreover the family $\mathbb{X}$ satisfies Brjuno's (ω) condition. Let $K \Subset \Omega$: there exists a neighborhood $U \subset D$ of O and a family of diffeomorphisms:*

$$\phi_\eta : U \to \mathbb{C}^n,$$

$\eta \in K$, such that $\phi_\eta(z) = w$ linearize $\mathbb{X}_\eta$:

$$D\phi_\eta \circ \phi_\eta^{-1} \mathbb{X}_\eta \circ \phi_\eta^{-1} = A_\eta.$$

Moreover, if $\phi(z, \eta) = \phi_\eta(z)$, then: $\phi : U \times \overline{K} \to T\mathbb{C}^n$ is continuous and it is holomorphic in $U \times K$.

The following extension of Taylor's theorem will be used in proving Theorem 3.1: it is classical, and its proof follows easily from the Straightening Out Theorem. Let us recall the classical definition of the push-forward operator:

$$\left(\phi_{\mathbb{U}}^t\right)_* \mathbb{X} = D\phi_{\mathbb{U}}^{-t} \circ \phi_{\mathbb{U}}^t \mathbb{X} \circ \phi_{\mathbb{U}}^t.$$

Lemma 3.1. *Let $\mathbb{X}$, $\mathbb{U}$ be two analytic vector fields, and let*

$$L_{\mathbb{U}} \mathbb{X} = [\mathbb{U}, \mathbb{X}], \qquad L_{\mathbb{U}}^{(m+1)} = L_{\mathbb{U}} L_{\mathbb{U}}^{(m)}.$$

Then, denoting $L_{\mathbb{U}}^{(0)} \mathbb{X} = \mathbb{X}$ there exists $R > 0$ such that for $|t| < R$:

$$\left(\phi_{\mathbb{U}}^t\right)_* \mathbb{X} = \sum_{k=0}^{\infty} \frac{t^k}{k!} L_{\mathbb{U}}^{(k)} \mathbb{X}. \tag{19}$$

We will frequently use the above expansion in the form:

$$\left(\phi_{\mathbb{U}}^{-t}\right)_* \mathbb{X} = \sum_{k=0}^{\infty} (-1)^k \frac{t^k}{k!} L_{\mathbb{U}}^{(k)} \mathbb{X}.$$

Proof of Theorem 3.1. (Based on [6], Theorem 3.) The proof will consist in the description of the iterative scheme leading to the definition of $\phi_\eta = \phi_\eta^\infty$ in the statement of the theorem as the limit of a sequence of analytic diffeomorphisms $\{\phi_\eta^{(N)}\}_{N=p}^\infty$ which linearize the original vector field up to order 2^k, showing the existence of a common domain D_{ρ_∞} of definition of these diffeomorphisms and their regular dependence on $\eta \in K$.

Throughout the proof a holomorphic function or a holomorphic vector field are *m-flat* if their first m derivatives at 0 are zero; moreover, emphasizing the algebraic point of view, two holomorphic vector fields U, V, both singular at 0, satisfy:

$$U(z) = V(z) \mod |z|^p$$

if and only if:

$$U(z) - V(z) = O\left(|z|^{p+1}\right).$$

The kth-step of the iteration deals with a vector field $\mathbb{X}_{\eta,k} = \mathbb{S}_\eta + \mathbb{N}_{\eta,k}$, decomposed in its semi-simple and nilpotent parts as described in the introduction. The nilpotent part $\mathbb{N}_{\eta,k}$ possibly contains the nilpotent part of the Jordan Normal Form of the linearization of X at 0, and a nonlinear part which is contained in $\chi^{>2^k}$, where

$$\chi^{>2^k} = \left\{\mathbb{X} \in \chi:\ \mathbb{X} \text{ is } 2^k\text{-flat at } O\right\},$$

and in z-coordinates the vector field $\mathbb{X}_{\eta,k}$ is analytic in the polydisk:

$$D_{\rho_k} = \left\{z \in \mathbb{C}^n:\ |z| < \rho_k\right\}$$

where $|z| = \max\{|z_j|:\ j = 1, \ldots, n\}$. The expression $X_{\eta,k}(z) = A_\eta z + N_{\eta,k}(z)$ of $\mathbb{X}_{\eta,k}$ in z-coordinates, where $A_\eta = S_\eta + B_\eta$ is the Jordan Normal Form of $\mathbb{A}_\eta = D\mathbb{X}_\eta(O)$, verifies:

$$S_\eta z = \lambda_1(\eta) z_1 \frac{\partial}{\partial z_1} + \cdots + \lambda_n(\eta) z_n \frac{\partial}{\partial z_n}, \tag{20}$$

$$N_{\eta,k}(z) = \mathcal{O}\big(|z|^{2^k+1}\big). \tag{21}$$

As the vector field is formally semi-simple: $B_\eta = 0$.

We look for an analytic change of coordinates $\phi_{\eta,k} : D_{\rho_k} \times K \to \mathbb{C}^n$:

$$\phi_{\eta,k} = \phi^{(-1)}_{\mathbb{U}_{\eta,k}} \tag{22}$$

where $\mathbb{U}_{\eta,k} : D_{\rho_k} \times K \to T\mathbb{C}^n$ verifies:

$$\mathbb{U}_{\eta,k} \in \chi^{>2^k} \tag{23}$$

and $\phi^{(-1)}_{\mathbb{U}_{\eta,k}}$ is its time-(-1) flow, such that if:

$$(\phi_{\eta,k})_* V(z) := d\phi^{-1}_{\eta,k}\big(\phi_{\eta,k}(z)\big) V\big(\phi_{\eta,k}(z)\big) \tag{24}$$

then

$$(\phi_{\eta,k})_* \mathbb{X}_{\eta,k} = \mathbb{A}_\eta + \mathbb{N}_{\eta,k+1}, \tag{25}$$

$$\mathbb{N}_{\eta,k+1}(z) \in \chi^{>2^{k+1}}. \tag{26}$$

The proof will begin with a formal part, where we will show that such vector field $\mathbb{U}_{\eta,k}$ exists as a polynomial vector field and satisfies (23), (25), (26). Firstly, we forget the question of existence of the time-(-1) map and prove that, if it exists, it satisfies (25), (26). In fact, from Lemma 3.1:

$$(\phi_{\eta,k})_*(\mathbb{A}_\eta + \mathbb{N}_{\eta,k}) = \mathbb{S}_\eta + \big\{\mathbb{N}_{\eta,k} - [\mathbb{U}_{\eta,k}, \mathbb{S}_\eta]\big\} + \left\{-[\mathbb{U}_{\eta,k}, \mathbb{N}_{\eta,k}] + \frac{1}{2}\big[\mathbb{U}_{\eta,k}, [\mathbb{U}_{\eta,k}, \mathbb{S}_\eta + \mathbb{N}_{\eta,k}]\big] + \cdots\right\}.$$

As $\mathbb{U}_{\eta,k}, \mathbb{N}_{\eta,k} \in \chi^{>2^k}$ we have:

$$\left\{-[\mathbb{U}_{\eta,k}, \mathbb{N}_{\eta,k}] + \frac{1}{2}\big[\mathbb{U}_{\eta,k}, [\mathbb{U}_{\eta,k}, \mathbb{S}_\eta + \mathbb{N}_{\eta,k}]\big] + \cdots\right\} \in \chi^{>2^{k+1}}$$

hence to find $\mathbb{U}_{\eta,k}$ which satisfies formally (25) we must prove, in z-coordinates:

$$N_{\eta,k}(z) - [U_{\eta,k}, S_\eta](z) = 0 \mod |z|^{2^{k+1}}. \tag{27}$$

Eq. (27) splits in $2^k + 1$ equations for the homogeneous parts of degrees $2^k, \ldots, 2^{k+1}$:

$$N^{(j)}_{\eta,k}(z) - \big[U^{(j)}_{\eta,k}, S_\eta\big](z) = 0 \quad \text{in } \mathbb{H}^{(j)}_n \tag{28}$$

where $\mathbb{H}^{(j)}_n$ is the vector space of homogeneous vector polynomials of degree j. If we look for a polynomial vector field:

$$U_{\eta,k}(z) = \sum_{j=2^k}^{2^{k+1}} U^{(j)}_{\eta,k}(z) \tag{29}$$

where $U^{(j)}_{\eta,k}(z) \in \mathbb{H}^{(j)}_n$, it is well known, see e.g. [6], that from the non-resonance hypothesis equation (28) admits a unique solution: in fact, the basic argument of Poincaré–Dulac Theorem is that if $A_\eta = S_\eta$, i.e. X_η is formally semi-simple, and S_η is diagonal in z-coordinates, with eigenvalues $\lambda_1(\eta), \ldots, \lambda_n(\eta)$, then the linear operator $X \to [S_\eta, X]$ on j-homogeneous vector fields is diagonal, too, in the base of j-homogeneous vector polynomials $z^{\underline{m}} \frac{\partial}{\partial z_l}$, $|\underline{m}| = j$, $l = 1, \ldots, n$, and its eigenvalues are the $\alpha(\underline{m}, l, \eta)$'s. In more detail, let

$$N^{(j)}_{\eta,k}(z) = \sum_{|\underline{m}|=j,\, l=1,\ldots,n} g_{\underline{m},l,\eta} z^{\underline{m}} \frac{\partial}{\partial z_l}$$

and

$$U^{(j)}_{\eta,k}(z) = \sum_{|\underline{m}|=j,\, l=1,\ldots,n} h_{\underline{m},l,\eta} z^{\underline{m}} \frac{\partial}{\partial z_l}$$

then

$$h_{\underline{m},l,\eta} = \frac{g_{\underline{m},l,\eta}}{\alpha(\underline{m},l,\eta)}. \tag{30}$$

This argument ends the proof of existence of the formal part of the iteration scheme.

Passing to the analytic part of the proof we must:

- prove that $\phi_{\eta,k}$ is well defined, i.e. prove that the flow $\phi^t_{\eta,k}$ is defined for every $z \in D_{\rho_k} \times \overline{D}_1$;
- prove that $\rho_k \to \rho_\infty$ and $\rho_\infty > 0$;
- prove that there exists $p \in \mathbb{N}$ such that if for every positive integer $N \geqslant p$ we define:

  $$\phi^{(N)}_\eta = \phi_{\eta,p} \circ \phi_{\eta,p+1} \circ \cdots \circ \phi_{\eta,N}$$

  then for every $z \in D_{\rho_\infty}$, $\eta \in K$ the sequences $\phi^{(N)}_\eta : D_{\rho_\infty} \to D_{\rho_p}$, $d\phi^{(N)}_\eta((\phi^{(N)}_\eta)^{-1}) : T_z\mathbb{C}^n \to T_z\mathbb{C}^n$, for $N \in \mathbb{N}$, converge uniformly in D_{ρ_∞} respectively to $\phi_{\infty,\eta} : D_{\rho_\infty} \to D_p$ and to $d\phi_{\infty,\eta}(\phi^{-1}_{\infty,\eta}) : T_z\mathbb{C}^n \to T_z\mathbb{C}^n$.

It will be easy to check that for every $\eta \in \Omega$, $\phi_{\infty,\eta}$ linearizes X_η, and that there exists a common nontrivial linearization disk if η varies in a compact set.

We begin this analytic part of the proof assuming the following inductive hypotheses, to be proved later: there exists $p \in \mathbb{N}$, such that for $k \geqslant p$ and $\eta \in K$:

$$\frac{1}{2} \leqslant \rho_k \leqslant 1, \tag{31}$$

$$|\mathbb{N}_{\eta,k}|_{\rho_k} \leqslant 1. \tag{32}$$

Here we defined the norm $|f|_{\rho_k}$ of a vector function f in the polydisk D_{ρ_k} as $|f|_{\rho_k} = \max\{|f_j|_{\rho_k}\colon\ j = 1,\dots,n\}$, $|f_j|_{\rho_k} = \sum_{\underline{l}} |c_{\underline{l}}|\rho_k^{|\underline{l}|}$, where $f_j(z) = \sum_{\underline{l}} c_{\underline{l}} z^{\underline{l}}$. Firstly we observe that, with a *una tantum* linear transformation we can obtain a vector field $X_{\eta,p}(z)$ which is holomorphic in D_1 and satisfies that:

$$X_{\eta,p}(z) = S_\eta z + N_{\eta,p}(z)$$

where $N_{\eta,p}(z)$ is a 2^p-flat vector field. To prove it, we recall that the Poincaré–Dulac Theorem allows to define local coordinates in a polydisk of radius ρ such that $X_{\eta,p}(z) = S_\eta z + R_\eta(z)$, where $R_\eta(z)$ is 2^p-flat. To obtain a vector field which is still a 2^p-flat perturbation of its linear part S_η, but is defined in a disk of radius 1, we consider the linear map:

$$\Psi_\lambda(z) = \lambda z = w$$

where $\lambda > 0$. Then

$$d\Psi_\lambda\left(\frac{w}{\lambda}\right)X_\eta\left(\frac{w}{\lambda}\right) = S_\eta w + \lambda R_\eta\left(\frac{w}{\lambda}\right).$$

Hence, choosing $\lambda = \rho_1^{-1}$:

$$\lambda\left|R_\eta\left(\frac{w}{\lambda}\right)\right|_1 = \lambda \sum_{|\underline{Q}|\geqslant 2^p+1} |R_{\underline{Q}}|\frac{1}{\lambda^{|\underline{Q}|}} = \frac{1}{\rho_1} \sum_{|\underline{Q}|\geqslant 2^p+1} |R_{\underline{Q}}|\rho_1^{|\underline{Q}|} = o(1)$$

when $\rho_1 \to 0$, therefore a suitable choice of ρ_1 and the application of the change of coordinates Ψ_{1/ρ_1} imply that $X_{\eta,p}$ is linearized up to 2^p-flat vector fields, is holomorphic in D_1 and satisfies (31), (32).

Hence we assume (31), (32) and $\rho_p = 1$ hold. Let $U_{\eta,k}$ be the solution of (27): from (30), (31) and the definition of ω_k in (17) we get:

$$|U_{\eta,k}|_{\rho_k} \leqslant \frac{1}{\omega_k}. \tag{33}$$

To improve this crude estimate we use flatness of $U_{\eta,k}$, homogeneity of the norm and a suitable slight reduction of the radius of the polydisk. If $r_k < \rho_k$:

$$|U_{\eta,k}|_{r_k} \leqslant \left(\frac{r_k}{\rho_k}\right)^{2^k} |U_{\eta,k}|_{\rho_k} \tag{34}$$

which together with (33) gives:

$$|U_{\eta,k}|_{r_k} \leqslant \left(\frac{r_k}{\rho_k}\right)^{2^k} \frac{1}{\omega_k}. \tag{35}$$

We observe that this estimate holds for every $\eta \in K$. The iteration scheme will use three polydisks: the radius ρ_k refers to the present step of iteration, ρ_{k+1} will be the radius of the polydisk of the next step, and r_k satisfying:

$$\rho_{k+1} < r_k < \rho_k$$

is the radius of an auxiliary polydisk used in the process. These radii are linked by

$$\tau_k = \frac{r_k}{\rho_k} = \left(\frac{\omega_k}{2^k}\right)^{\frac{1}{2^k}}, \tag{36}$$

$$\sigma_k = \frac{\rho_{k+1}}{\rho_k} = \left(\frac{\omega_k}{2^{2k}}\right)^{\frac{1}{2^k}}. \tag{37}$$

We remark that τ_k's and σ_k's do not depend on $\eta \in K$. From (35):

$$|U_{\eta,k}|_{r_k} \leqslant \tau_k^{2^k} \frac{1}{\omega_k} = \frac{1}{2^k}. \tag{38}$$

The flow $(t,z) \to \phi^t_{U_{\eta,k}}(z) = \phi^t_{\eta,k}$ is analytic for $|t|$, $|z|$ sufficiently small: we are going to prove that it is defined for $|t| \leqslant 1$ and $|z| < \rho_{k+1}$. From

$$\phi^t_{\eta,k}(z) = z + \int_0^t U_{\eta,k}\big(\phi^s_{\eta,k}(z)\big)\,ds$$

and from (31), (32) (38), if

$$\rho_{k+1} + \frac{1}{2^k} < r_k \tag{39}$$

we have that if $z \in D_{\rho_{k+1}}$ then $\phi^t_{\eta,k}(z) \in D_{r_k}$ for every $|t| \leqslant 1$. We will prove shortly that (39) follows from (36). We are going to show now how these arguments lead to the end of the proof. From the definition of $\phi_{\eta,k}$ given in (22), let

$$\phi^{(N)}_\eta = \phi_{\eta,p} \circ \phi_{\eta,p+1} \circ \cdots \circ \phi_{\eta,N}.$$

To give this definition analytic sense we shall prove that, independently of $\eta \in K$:

$$\rho_k \downarrow \rho_\infty > 0 \tag{40}$$

hence for every positive integer N:

$$\phi^{(N)}_\eta : D_{\rho_\infty} \to D_1.$$

Let us remark that if $z \in D_{\rho_\infty}$ then, related to (22), (24) we have:

$$d\big(\phi^{(N)}_\eta\big)^{-1}\big(\phi^{(N)}_\eta\big) : T_z\mathbb{C}^n \to T_z\mathbb{C}^n$$

and convergence of the two lastly defined sequences is the goal of the rest of the proof.

Now we are going to prove the inductive hypotheses (31), (32), the inequalities (39) and (40). Let us begin from this last property. From $\rho_p = 1$ we have, formally:

$$\prod_{k=p}^{\infty} \sigma_k = \lim_{m\to\infty} \rho_m = \rho_\infty$$

therefore we must prove convergence of:

$$\prod_{k=p}^{\infty} \sigma_k = \prod_{k=p}^{\infty} \left(\frac{\omega_k}{2^{2k}}\right)^{\frac{1}{2^k}}$$

which is equivalent to Brjuno's condition (ω): hence (40) follows. Let us prove now (39), namely that putting $\rho = \rho_k$:

$$\rho\left(\left(\frac{\omega_k}{2^k}\right)^{\frac{1}{2^k}} - \left(\frac{\omega_k}{2^{2k}}\right)^{\frac{1}{2^k}}\right) > \frac{1}{2^k} \tag{41}$$

where from (31): $\frac{1}{2} \leqslant \rho \leqslant 1$. Writing (41) as:

$$\rho > \frac{1}{\omega_k^{\frac{1}{2^k}} 2^{k(1-\frac{2}{2^k})}(e^{\frac{k\log 2}{2^k}}-1)}$$

and observing that from Brjuno's condition (ω) $\omega_k^{\frac{1}{2^k}} \to 1$, it is sufficient to prove that for $\rho \in [\frac{1}{2}, 1]$ and $k \geqslant p$, p sufficiently big:

$$\rho > \frac{1}{2^{k(1-\frac{2}{2^k})}(e^{\frac{k\log 2}{2^k}}-1)}. \tag{42}$$

From elementary considerations:

$$\frac{2^k}{(2^k)^{\frac{2}{2^k}}}\left(e^{\frac{k\log 2}{2^k}}-1\right) = \frac{k}{1+o(1)}\left(2\log 2 + \log 2\mathcal{O}\left(\frac{\log 2^k}{2^k}\right)\right) \to \infty$$

as $k \to \infty$, and this ends the proof of (39).

We prove now (31), (32).

Firstly, the inequality on ρ_k follows from $\rho_k = \sigma_k\sigma_{k-1}\cdots\sigma_p$, convergence of the infinite product and a suitable choice of p. Here we observe that the definition of p depends on the convergence of the infinite product involving the ω_k's, and on the definition of parameter $\lambda = \lambda(\eta)$ in the linear transformation Ψ_λ. From Brjuno's condition (ω) for the family $\mathbb{X}$ and the continuous dependence of $\lambda(\eta)$ on the parameter and relative compactness of K we can conclude that there exists p independent of $\eta \in K$ satisfying the above requests.

Let us prove that:

$$|N_{\eta,k}|_{\rho_k} < 1$$

implies that:

$$|N_{\eta,k+1}|_{\rho_{k+1}} < 1.$$

We recall that:

$$X_{\eta,k+1}(z) = S_\eta z + N_{\eta,k+1}(z) = \left(\phi_{\eta,k}^{-1}\right)_*(S_\eta + N_{\eta,k})(z) - S_\eta z$$

hence:

$$N_{\eta,k+1}(z) = \left[\left(\phi_{\eta,k}^{-1}\right)_* S_\eta - S_\eta\right](z) + \left(\phi_{\eta,k}^{-1}\right)_* N_{\eta,k}(z). \tag{43}$$

To estimate the norms in the previous equality when $|z| < \rho_{k+1}$ we use a dynamical approach based on:

$$\phi_{\eta,k}^t(z) = z + \int_0^t U_{\eta,k}\left(\phi_{\eta,k}^s(z)\right) ds$$

and its consequence:

$$D\phi_{\eta,k}^t(z) = E_n + \int_0^t DU_{\eta,k}\left(\phi_{\eta,k}^s(z)\right) D\phi_{\eta,k}^s(z)\, ds \tag{44}$$

where E_n is the identity in $\mathbb{C}^n$ and from (39) the equation in (44) holds for $|t| \leqslant 1$. From (39) and (38) $|U_{\eta,k}(\phi_{\eta,k}^s(z))| < \frac{1}{2^k}$ and this leads to the following fundamental estimate on the differential $DU_{\eta,k}(\phi_{\eta,k}^s(z))$. We recall that:

$$U_{\eta,k,j}(z) = \sum_{2^k+1 \leqslant |\underline{Q}| \leqslant 2^{k+1}} h_{\underline{Q},j} z^{\underline{Q}}$$

where $\underline{Q} = (q_1, \ldots, q_n)$, implying that:

$$\frac{\partial U_{\eta,k,j}(z)}{\partial z_l} = \sum_{2^k+1 \leqslant |\underline{Q}| \leqslant 2^{k+1}} q_l h_{\underline{Q},j} z^{\underline{Q}-\underline{e}_l},$$

$\underline{e}_l$ being the lth-element of the standard basis in $\mathbb{C}^n$. Then

$$\left|\frac{\partial U_{\eta,k,j}(z)}{\partial z_l}\right|_{r_k} \leqslant \frac{2^{k+1}}{r_k} \sum_{2^k+1\leqslant|\underline{Q}|\leqslant 2^{k+1}} |h_{\underline{Q},j}| r_k^{|\underline{Q}|} \leqslant \frac{2^{k+1}}{r_k 2^k} = \frac{2}{r_k}.$$

From (44) and the above inequality we get, for $|t| \leqslant 1$:

$$\left|D\phi^t_{\eta,k}(z)\right|_{\rho_{k+1}} \leqslant 1 + \int_0^1 \frac{2}{r_k}\left|D\phi^s_{\eta,k}(z)\right|_{\rho_{k+1}} ds \tag{45}$$

and therefore from (38), from the fact that $r_k \to \rho_\infty \geqslant \frac{1}{2}$ and from Gronwall inequality we obtain, for a suitable choice of p:

$$\left|N_{\eta,k+1}(z)\right|_{\rho_{k+1}} \leqslant \left(e^5+1\right)|S_\eta| + e^5\left|N_{\eta,k}(z)\right|_{\rho_{k+1}}. \tag{46}$$

The frequently used homogeneity property and the inductive hypothesis $|N_{\eta,k}(z)|_{\rho_k} < 1$ implies that $|N_{\eta,k}(z)|_{\rho_{k+1}} = o(1)$ as $k \to \infty$, hence with a suitable choice of p we get that $|N_{\eta,k}(z)|_{\rho_{k+1}} < \frac{1}{2e^5}$. On the other hand it is clear that the action of $\mathbb{C}^*$ by multiplication:

$$(\mu, X) \to \mu X$$

should have no effects on the linearizability of vector fields, hence has no effect on the choice of $U_{\eta,k}$, too, as follows immediately from the independence on μ of the definition in (30). This remark implies that we can always suppose that, up to multiplication for a suitable positive constant:

$$\left(e^5+1\right)|S_\eta| < \frac{1}{2}$$

and we conclude from (46) that the estimate:

$$|N_{\eta,k+1}|_{\rho_{k+1}} < 1$$

does hold.

Now convergence of a subsequence of $\phi^{(N)}_\eta : D_{\rho_\infty} \to D_1$, $N \geqslant p$, to $\phi_{\infty,\eta} : D_{\rho_\infty} \to D_1$ follows from Montel's Theorem, and from Weierstrass's Theorem along such subsequence $d\phi^{(N)}_\eta(w) \to d\phi_{\infty,\eta}(w)$ uniformly for $w \in \overline{D}_\rho$, for any $\rho < \rho_\infty$. As $d\phi^{(N)}_\eta(0) = d\phi_{\infty,\eta}(0) = E_n$ all the $d\phi^{(N)}_\eta(w)$ are invertible for $|w| < \rho_0$ for a suitable $\rho_0 > 0$, and by straightforward computation:

$$d\left(\phi^{(N)}_\eta\right)^{-1}\left(\phi^{(N)}_\eta(z)\right)X\left(\phi^{(N)}_\eta(z)\right) = \left(\phi^{-1}_{\eta,N}\right)_* \circ \cdots \circ \left(\phi^{-1}_{\eta,p}\right)_*$$

hence $\phi_{\infty,\eta} : D_{\rho_\infty} \to D_1$ linearizes X in a neighborhood of 0 contained in D_{ρ_∞}; moreover the convergence just described is convergence of the whole sequence $\{\phi^{(N)}_{eta}\}^\infty_{N=p}$, as a consequence of uniqueness of the formal linearization in absence of resonance relations. Finally, the fact that there exists a common domain of invertibility of the linearizing transformations when η varies in a compact set K is an easy consequence of the Inverse Function Theorem, as well as the analytic dependence of the inverse function on the parameter η. □

We present now an example of application of Theorem 3.1: we consider the simplest situation, namely the case of an analytic family of vector fields in a neighborhood of the origin in $\mathbb{C}^2$. We consider the family of holomorphic vector fields:

$$X_\eta(z) = z_1\frac{\partial}{\partial z_1} + \lambda(\eta) z_2 \frac{\partial}{\partial z_2} + F_\eta(z) \tag{47}$$

where $F_\eta(z) = \mathcal{O}(|z|^2)$ and $\eta \to X_\eta$ is analytic for $\eta \in \Omega$, continuous in $\overline{\Omega}$, with $\eta_0 \in \partial\Omega$ and, for $\varepsilon > 0$, $0 \leqslant \delta < \frac{\pi}{2}$:

$$\lambda(\eta) \in \mathcal{S}^+_{\varepsilon,\delta}\left(\lambda(\eta_0)\right) = \lambda(\eta_0) + \left\{\rho e^{i(\frac{\pi}{2}-\theta)}\colon 0 \leqslant \rho < \varepsilon,\ |\theta| < \delta\right\}. \tag{48}$$

An analogous definition could be given in the case when the ratio of the eigenvalues of the differential of the vector fields at the origin varies in the sector $\mathcal{S}^-_{\varepsilon,\delta}(\lambda(\eta_0))$, where with respect to (48) $\frac{\pi}{2}$ is substituted by $-\frac{\pi}{2}$. We will suppose that

$$\left(\lambda(\eta_0)\right) \in \mathbb{R}^- \backslash \{\mathbb{Q}\} \tag{49}$$

and the origin O is a singular point of Siegel type for $X_{\eta_0}(z)$, which satisfies Bruno's condition. It is worth commenting (47), (48) and the above hypotheses: (47) is a rather general form for a family of holomorphic vector fields in $\mathbb{C}^2$, for the geometric properties of the family do not depend on factors $\mu(\eta) \in \mathbb{C}^*$; the choice of the sector $\mathcal{S}^+_{\varepsilon,\delta}(\lambda(\eta_0))$ is motivated from the fact that if $\lambda(\eta_0) \notin \mathbb{R}^-$ the singular points at O of the fields of the family are in the Poincaré domain and the description of the analytic dependence on the parameters of the linearizing maps has been given in the previous section. The choice of the vertex of the sector $\mathcal{S}^+_{\varepsilon,\delta}(\lambda(\eta_0))$ allows application of the results in Theorem 3.1, while the shape of the sector does not allow $\lambda(\eta)$ varies in $\mathbb{R}$: in such case even topological stability of the local foliation defined by the fields of the family is known to be false, see [8].

Theorem 3.2. *Let $X : D \times \overline{\mathcal{S}}^{+}_{\varepsilon,\delta}(\lambda(\eta_0)) \to T'\mathbb{C}^n$ be continuous and satisfy* (47), (48); *let $X : D \times \mathcal{S}^{+}_{\varepsilon,\delta}(\lambda(\eta_0)) \to T'\mathbb{C}^n$ be holomorphic. Then there exists a neighborhood U of $0 \in \mathbb{C}^2$ and an analytic family*

$$\phi : U \times \left(\mathcal{S}^{+}_{\varepsilon,\delta}\big(\lambda(\eta_0)\big)\backslash\{\eta_0\}\right) \to \mathbb{C}^2$$

continuous up to η_0, such that for every fixed η $\phi_\eta(\cdot) = \phi(\cdot,\eta)$ is a diffeomorphism in U, and

$$\big((\phi_\eta)_* X_\eta\big)(z) = z_1 \frac{\partial}{\partial z_1} + \lambda(\eta) z_2 \frac{\partial}{\partial z_2}.$$

Proof. Firstly, we observe that no resonance relation holds for any vector field of the family. From Theorem 3.1 it is sufficient to prove that $\{X_\eta\}_{\eta\in\Omega}$ is an analytic family satisfying Bruno's condition (ω). We recall that, adapting the definition of $\omega_k(\eta)$ to the 2-dimensional case:

$$\omega_k(\eta) = \inf\big\{\big|p + q\lambda(\eta)\big|:\ p,q \in \mathbb{N}\cup\{0,-1\},\ 1 \leqslant p+q < 2^{k+1}\big\}.$$

Denoting:

$$\alpha(p,q,\eta) = p + q\lambda(\eta) = \alpha(p,q,\eta_0) + q\rho e^{i(\frac{\pi}{2}-\theta)} \tag{50}$$

from elementary geometric considerations we have:

$$\big|\alpha(p,q,\eta)\big| \geqslant \big|\alpha(p,q,\eta_0)\big| \cos\delta.$$

In fact, $\alpha(p,q,\eta_0) \in \mathbb{R}$ and $\alpha(p,q,\eta)$ belongs to the cone $\mathcal{C} = \{z \in \mathbb{C}:\ z = \alpha(p,q,\eta_0) + \rho e^{i(\frac{\pi}{2}-\theta)},\ |\theta| < \delta,\ \rho \geqslant 0\}$, hence it has distance from the origin in $\mathbb{C}$ greater than $|\alpha(p,q,\eta_0)|\cos\delta$. Finally for every $\eta \in \Omega$:

$$C = \omega_1 = \omega_1(\eta_0)\cos\delta \geqslant \omega_k$$

for every $k \in \mathbb{N}$. Therefore $\{\omega_k\}_k$ satisfies Bruno's condition (ω), and the application of Theorem 3.1 ends the proof. □

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